AY C10/L&S C70U **Name:** ____________________
Fall 2018
Worksheet 4
09/18/2018

This worksheet covers the basics of Kepler's laws and orbital mechanics. There are 4 pages (including this cover page) and 2 questions, with 70 possible points. Accuracy is definitely desired, but effort and clear physical reasoning are far more important than the final answer, *especially* for the challenge problems.

After extensive observations of the Solar System over decades, the German astronomer Johannes Kepler found that the planets followed three major laws of orbital motion, namely (1) all planets follow elliptical trajectories around the Sun, with the Sun at one focus; (2) All planets sweep out equal areas of their orbits in equal times, and (3) the square of the period is proportional to the cube of the semimajor axis (half the length of the "long side" of an ellipse).

For a long time, Kepler's laws were seen as a set of scaling relations that just happened to work. Newton's theory of universal gravitation, however, naturally explained and generalized Kepler's laws to any set of orbiting bodies. Under Newtonian gravity, any two bodies orbit around their common center of mass rather than the more massive object—for the Solar System, however, this is essentially the position of the Sun because $M_{\rm Sun} \gg M_{\rm planets}$. Newton's work led to the following equations:

$$T = 2\pi \left(\frac{a^3}{G(m_1 + m_2)} \right)^{1/2} \tag{1}$$

Ultimately, Keplerian orbital mechanics is driven by the conservation of orbital energy, linear momentum, and angular momentum. Studying these connections quantitatively is *far* beyond the scope of this course—I am more than happy to derive them in office hours, however.

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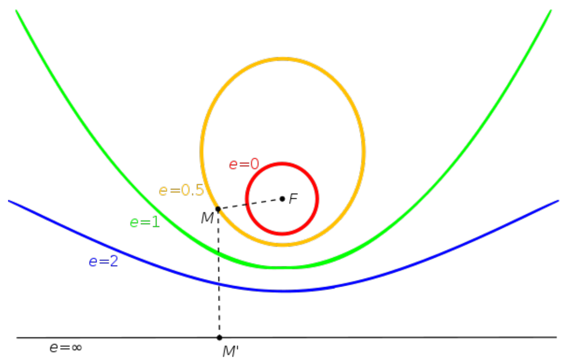
1. (40 points) In this question, your UGSI permits you to have some fun with Kepler's three laws.
   (a) (10 points) **State and explain** Kepler's three laws in your own words, to the best of your ability.
      1. **All planets orbit the Sun in ellipses, with the Sun at one focus**—that means that planets make one closest and one farthest approach over the course of an orbit.
      2. **Planets sweep out equal areas in equal times**—as described in section, this implies that a planet will move faster at its closest approach than at its farthest.
      3. **The square of the period is proportional to the cube of the semimajor axis**—plugging in constants, this implies that the period of an orbit $T_{\rm orb} = 2\pi\sqrt{a^3/GM_\odot}$.

(b) (15 points) Kepler's First Law—the statement that "planets orbit in ellipses with the Sun at one focus"—can be expressed in the following equation:

$$r(\theta) = \frac{d}{1 + \epsilon\cos(\theta)} \tag{2}$$

Where the Sun is at the origin. In this subpart, we analyze some implications of this expression, and its connection to Kepler's other laws.

i. (5 points) **Make sketches** of this equation, for $\epsilon = 0$, $\epsilon = 0.5$ and $\epsilon = 1$. You may use the graphing program of your choice. **What similarities and differences** do you notice between these cases?



The orbit with $\epsilon = 0$ is perfectly circular, while that for $\epsilon = 0.5$ is somewhat elongated. The orbit for $\epsilon = 1$ is a parabolic orbit, the limiting case of an ellipse—the farthest approach of the planet (apoapsis) is located at infinity, where the planet stops. (*Note: The $\epsilon = 2$ orbit is a* hyperbolic *orbit, for which it would have some velocity even at infinity.*)

ii. (7 points) The *semimajor axis* is just the distance at closest approach (periapsis) plus the distance at farthest approach (apoapsis) divided by two. **Compute the orbital period** as a function of semimajor axis. This can be computed by an application of Kepler's Third Law:

$$T_{\text{orb}} = 2\pi\sqrt{\frac{a^3}{GM}} \tag{3}$$

Because the mass of the planet is much less than that of the star, and it is in the denominator, we can omit it.

iii. (3 points) If $\epsilon = 0$, the orbit is a perfect circle. In this special case, **compute the speed** of the orbiting body.
The speed of an orbit is simply the distance that the planet travels over the time that it takes for the planet to orbit. Because the orbit is circular, we know by Kepler's second law that the speed must be a constant, and that the distance around it is $2\pi a$.

$$v = \frac{D_{\text{orb}}}{T_{\text{orb}}} = \frac{2\pi a}{2\pi(a^3/GM)^{1/2}} = \sqrt{\frac{GM}{a}} \tag{4}$$

which is the standard formula for orbital velocity.

(c) (15 points) In this subpart, we study the implications of Kepler's Second Law.

i. (3 points) **Compute the length and area of an arc** of radius r and angle $\delta\theta$. Make sure to state the angle in radians.

The length of the arc is $r\delta\theta$ and the area of the resulting slice is $r^2\delta\theta/2$, if the angle $\delta\theta$ is expressed in radians.

ii. (6 points) Kepler's Second Law can be restated as "the area swept out by an orbit is proportional to the time taken to sweep out that area." Using the equation in part (b), **compute the length and area of two arcs of small angle** $\delta\theta$, one where the arc radius is the closest-approach distance $(d/(1+\epsilon))$ and another where it is the farthest-approach distance $(d/(1-\epsilon))$. **What is the ratio of time taken to travel an angle** $\delta\theta$ between the closest-approach and farthest-approach distance?

From the statement of the problem, the time required to sweep an area is proportional to that area. We can state this as

$$T_{\text{sweep}} \propto A_{\text{sweep}} \tag{5}$$

This can be evaluated at two times, T_1 and T_2. By taking the ratio of the two, the proportionality becomes an equality because the implied constant divides out.

$$\frac{T_1}{T_2} = \frac{A_1}{A_2} \tag{6}$$

Substituting in expressions for the areas, we get that

$$\frac{T_{\text{closest}}}{T_{\text{farthest}}} = \frac{(d/(1+\epsilon))^2\delta\theta}{(d/(1+\epsilon))^2\delta\theta} = \left(\frac{1-\epsilon}{1+\epsilon}\right)^2 \tag{7}$$

iii. (6 points) Using your result from the previous part, **compute the ratio of speeds** between the closest and farthest approach of a planet to the Sun. **What trend do you note** when you test $\epsilon = 0$ ("circular"), $\epsilon = 0.5$ ("elliptical"), and $\epsilon = 1$ ("unbound") orbits?

The ratio of speeds $v_{\text{closest}}/v_{\text{farthest}}$ cam be computed using

$$\frac{v_{\text{closest}}}{v_{\text{farthest}}} = \frac{d_{\text{closest}}/T_{\text{closest}}}{d_{\text{farthest}}/T_{\text{farthest}}} = \frac{d_{\text{closest}}}{d_{\text{farthest}}}\left(\frac{T_{\text{closest}}}{T_{\text{farthest}}}\right)^{-1} \tag{8}$$

Substituting in what we found in the previous equation, and that $d_{\text{closest}} = d\delta\theta/(1+\epsilon)$ and $d_{\text{farthest}} = d\delta\theta/(1-\epsilon)$, we can show that

$$\begin{aligned}\frac{v_{\text{closest}}}{v_{\text{farthest}}} &= \frac{d\delta\theta/(1+\epsilon)}{d\delta\theta/(1-\epsilon)}\left(\frac{1-\epsilon}{1+\epsilon}\right)^{-2} = \left(\frac{1+\epsilon}{1-\epsilon}\right)^{-1}\left(\frac{1+\epsilon}{1-\epsilon}\right)^{2} \\ &= \frac{1+\epsilon}{1-\epsilon}\end{aligned} \tag{9}$$

2. (30 points) *Challenge Problem* a star of mass M_* is orbiting a black hole of mass $M_{\rm BH}$ in deep space—*we don't know the specific values of these masses,* only that $M_{\rm BH} \gg M_*$. We can infer the orbital period $T_{\rm orb}$ of the system because it flickers every time the star disappears behind the black hole; however, we cannot resolve the two bodies and compute an angular separation.

   Using the information we know and Kepler's Third Law, **could we compute the semimajor axis of the star's orbit**? If not, **what additional piece of information** would we need, and **how could we obtain it observationally**?